Determination of the Optimal Elliptical Trajectories Around the Earth and Moon

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Abstract

Current space exploration programs call for the establishment of a permanent Human presence on the Moon. This paper considers periodic orbits of a shuttle between the Earth and the Moon. Such a shuttle will be needed to bring supplies to the Moon outpost and carry back those resources that are in short supply on Earth. To keep this shuttle in permanent periodic orbit it must have a thruster that forces it into an elliptical orbit from perigee near Earth to an apogee just beyond the Moon and back to perigee. The impacts of the Earth, Moon and Sun gravity on this orbit are considered. For this model we determine the eccentricity that minimizes the thrust requirements and the lunar Δv requirements. We show that optimal placements of the eccentricity of the shuttle orbit can produce significant improvement in thrust (and fuel) requirements.

1 Introduction

Over the years there have been many suggested trajectories from the vicinity of the Earth to the vicinity of the Moon[1, 2, 5, 6, 8, 13, 15]. Extensive research is ongoing also on orbits and trajectories in more general contexts.[3, 4, 7, 9, 10, 11, 12]

The Apollo missions have shown that free-return ballistic figure eight shaped trajectories have required only minor orbit adjustments to traverse between Earth and Moon[14]. However such trajectories provide only for one round trip between the Earth and the Moon. In view of current plans to establish permanent human presence on the Moon it is appropriate to consider a shuttle in a periodic orbit in between the Earth and the Moon. In the present paper we consider elliptical orbits for such a shuttle (This problem was considered briefly in [8].). We are assuming that the shuttle is equipped with a thruster that propels it in a planar elliptical orbit from a perigee near Earth to an apogee beyond the moon and back to perigee. The thrust negates the out-of-plane effects of solar gravity. The objective of this paper is to find the eccentricity of the elliptical orbit that minimizes the thrust and the lunar Δv requirements. It should determine the most efficient of all elliptic orbits for this shuttle around the Earth and Moon taking into account the Earth Moon and Sun gravity on the shuttle orbit.

Complementary studies are assumed. It is likely that a separate stage will be required for the energy needed to enter the elliptical transfer orbit. For this reason the initial Δv will constitute a separate study not included herein, although we will include a preliminary study of the eccentricity effects on the lunar Δv. Furthermore we neglect the variation of the Moon orbit from circular as well as Earth oblateness and solar radiation of the Sun. These effects can be considered in more refined studies or compensated through a control system.

Relevant standard data involving the Earth, Moon and Sun are available in [16]

2 The Orbit Requirements

This preliminary analysis will determine the eccentricity of the orbit which optimizes the maximum thrust requirement and also the Δv requirement needed to traverse between the Moon and the spacecraft orbit. This section presents a study of each. The $\Delta\, v$ requirement to traverse from Earth or Earth orbit to the elliptical orbit is also highly important but is not included in the present study. This activity will likely need a separate stage and a separate thruster thus requiring another study.

2.1 Angular Velocity and Time in Orbit

Fig 1 depicts a schematic view of the orbit of a spacecraft where the thrust forces a planar elliptical orbit in a frame fixed in the axis through the Earth and the Moon and rotating with the angular velocity ω of the Moon about the Earth. Relevant points on the Earth-Moon axis as measured from the Earth's center are as follows: The position of the Moon is x_M, the perigee of the orbit is x_p, the Earth-Moon center of gravity is x_c, the focus of the elliptical orbit is x_o, and x_a is the apogee. An arbitrary point on the orbit is denoted by the polar pair (r, θ) as measured from the focus. It follows that

$$r(\theta) = \frac{p}{1 + e\cos(\theta)}. \tag{2.1}$$

We denote by $x(\theta)$ the projection of $r(\theta)$ on the Earth-Moon axis as measured from the center of the Earth. We observe that $r(\theta)\cos(\theta)$ is negative for $\frac{\pi}{2} < \theta < \pi$. We can describe $x(\theta)$ as follows:

$$x(\theta) = x_0 - \frac{p\cos(\theta)}{1 + e\cos(\theta)} \tag{2.2}$$

where $x(0) = -x_p$ and $x(\pi) = x_a$. It follows that

$$x_p + x_0 = \frac{p}{1 + e}, \tag{2.3}$$

$$x_a - x_0 = \frac{p}{1 - e}. \tag{2.4}$$

Solving these equations for p and x_0 we obtain

$$p = \frac{x_a + x_p}{2}(1 - e^2) \tag{2.5}$$

$$x_0 = \frac{x_a - x_p}{2} - e\frac{x_a + x_p}{2}. \tag{2.6}$$

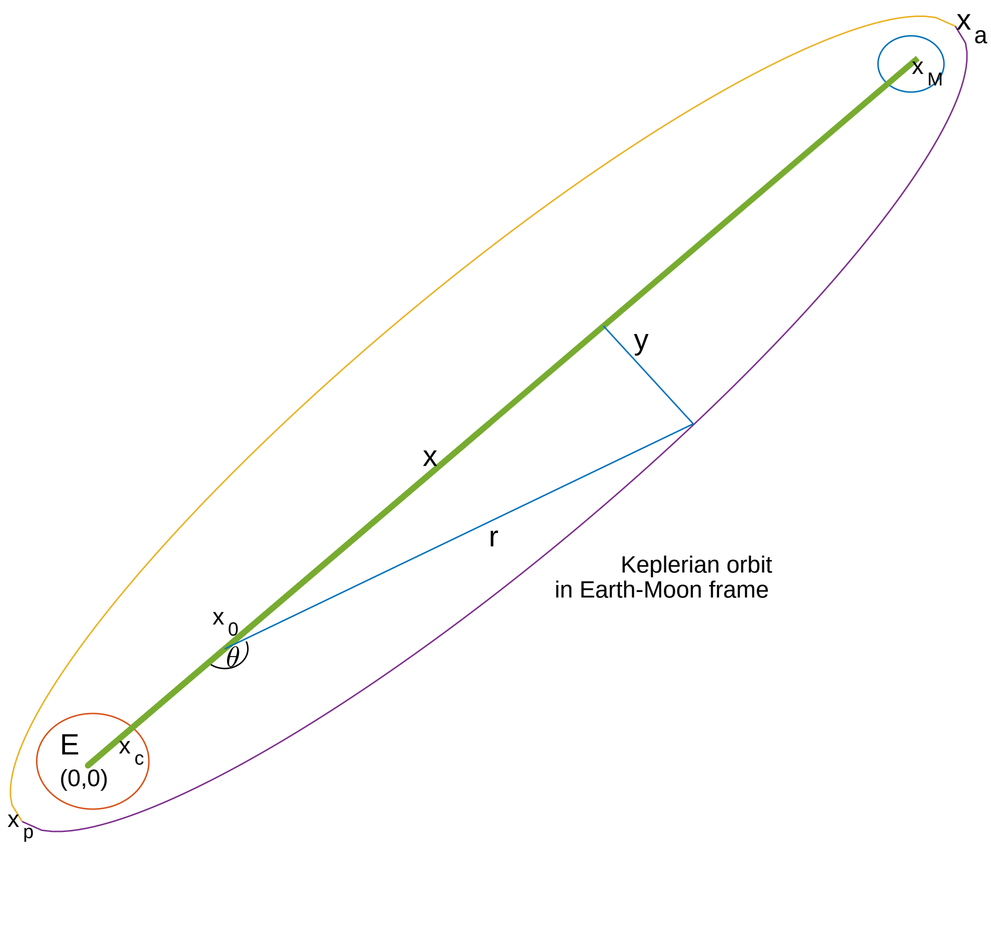

Figure 1: A schematic plot of the elliptical orbit about the Earth and Moon.

At the angle θ the position of the spacecraft (see Figure 1) is determined by the vector

$\mathbf{r}(\theta) = (x(\theta), y(\theta))$ where $x(\theta)$ is given by (2.2) and

$$y(\theta) = \frac{p \sin\theta}{1 + e\cos\theta}. \tag{2.7}$$

It follows from (2.2) and (2.7) that

$$\dot{x}(\theta) = \frac{p\dot{\theta}\sin\theta}{(1 + e\cos\theta)^2}, \tag{2.8}$$

$$\dot{y}(\theta) = \frac{p\dot{\theta}(e + \cos\theta)}{(1 + e\cos\theta)^2}. \tag{2.9}$$

The angular velocity function $\dot{\theta}$ is defined as a function of θ on the interval $0 \leq \theta \leq 2\pi$. We may choose this function $\dot{\theta}$. The following properties are important in making this choice of the curve $\dot{\theta}(\theta)$. We require that it will be smooth, simple, and symmetric about $\theta = \pi$. On the basis of these criteria, we select the quadratic function:

$$\dot{\theta}(\theta) = -\frac{\dot{\theta}(\pi) - \dot{\theta}(0)}{\pi^2}(\theta - \pi)^2 + \dot{\theta}(\pi). \tag{2.10}$$

We require that $\dot{\theta}(0) \neq 0$ and $\dot{\theta}(\pi) \neq 0$. It follows that

$$\ddot{\theta}(\theta) = -\frac{2(\dot{\theta}(\pi) - \dot{\theta}(0))}{\pi^2}(\theta - \pi). \tag{2.11}$$

These expressions are completely determined by $\dot{\theta}(0)$ and $\dot{\theta}(\pi)$. Since $\dot{y}(0) = v_p$ and $\dot{y}(\pi) = v_a$ they can be gotten from (2.9) and (2.5):

$$\dot{\theta}(0) = \frac{(1 + e)v_p}{p} = \frac{2v_p}{(1 - e)(x_p + x_a)} \tag{2.12}$$

$$\dot{\theta}(\pi) = \frac{(1 - e)v_a}{p} = \frac{2v_a}{(1 + e)(x_p + x_a)}. \tag{2.13}$$

Time in orbit can be determined from (2.10) through the integral

$$t(\theta) = \frac{1}{\dot{\theta}(\pi)} \int_0^\theta \frac{ds}{1 + \frac{k}{\pi^2}(s - \pi)^2} \tag{2.14}$$

where

$$k = \frac{v_p(1 + e)}{v_a(1 - e)} - 1. \tag{2.15}$$

If $-1 < k < 0$ then the integral is evaluated as

$$t(\theta) = \frac{\pi}{2\sqrt{-k}\dot{\theta}(\pi)} \left[\ln \left| \frac{1 + \frac{\sqrt{-k}}{\pi}(\theta - \pi)}{1 - \frac{\sqrt{-k}}{\pi}(\theta - \pi)} \right| + \ln \left| \frac{1 + \sqrt{-k}}{1 - \sqrt{-k}} \right| \right], \tag{2.16}$$

if $k = 0$, it becomes

$$t(\theta) = \frac{\theta}{\dot{\theta}(\pi)}, \tag{2.17}$$

and if $k > 0$

$$t(\theta) = \frac{\pi}{\sqrt{k}\dot{\theta}(\pi)} \left[Arctan \frac{\sqrt{k}(\theta - \pi)}{\pi} + Arctan(\sqrt{k}) \right]. \tag{2.18}$$

The time t_f to traverse from perigee to apogee is therefore obtained from these expressions by setting $\theta = \pi$.

2.2 Gravitational Effects and Total Thrust

We transfer from the coordinate system through the Earth and the Moon that has been rotated through the angle ωt as depicted in Figure 1 to the original system at $t = 0$. Through a coordinate rotation, we obtain

$$X = -y(\theta) \sin \omega t + x(\theta) \cos \omega t \tag{2.19}$$

$$Y = y(\theta) \cos \omega t + x(\theta) \sin \omega t \tag{2.20}$$

where X and Y are measured in the original coordinate system at $t = 0$.

We observe that the components of the Earth and Moon gravitational forces in the the X-direction respectively are

$$F_{E_X}(\theta) = \frac{-g_E X}{[X^2 + Y^2]^{3/2}}, \tag{2.21}$$

$$F_{M_X}(\theta) = -\frac{g_M(x_M - X)}{[(x_M - X)^2 + Y^2]^{3/2}}, \tag{2.22}$$

and in the Y-direction

$$F_{E_Y}(\theta) = \frac{-g_E Y}{[X^2 + Y^2]^{3/2}}, \tag{2.23}$$

$$F_{M_Y}(\theta) = \frac{-g_M Y}{[(x_M - X)^2 + Y^2]^{3/2}}, \tag{2.24}$$

where g_E and g_M denote respectively the gravitational constants of the Earth and the Moon.

The gravitational force of the Sun on the spacecraft is written as

$$F_S = -\frac{g_S}{r_S^2} \tag{2.25}$$

where g_S is the gravitational constant of the Sun and r_S is the distance of the orbital plane of the Earth and Moon from the Sun neglecting the variation of the distance over the orbit of the spacecraft. The angle between the Sun and a normal to the orbital plane is denoted by γ. The out-of-plane component of the solar gravitation on the spacecraft is

$$F_{S_Z} = -\frac{g_S \cos\gamma}{r_S^2}. \tag{2.26}$$

The components of the solar gravitation in the X-direction and Y-direction respectively are

$$F_{S_X}(\theta) = -\frac{g_S \sin\gamma X}{r_s^2[X^2+Y^2]^{1/2}}, \tag{2.27}$$

$$F_{S_Y}(\theta) = -\frac{g_S \sin\gamma Y}{r_S^2[X^2+Y^2]^{1/2}}, \tag{2.28}$$

where X and Y are given respectively by (2.19) and (2.20).

Denoting the total thrust by $\mathbf{T}$, its components respectively are

$$T_X = \ddot{X} - F_{E_X}(\theta) - F_{M_X}(\theta) - F_{S_X}(\theta), \tag{2.29}$$

$$T_Y = \ddot{Y} - F_{E_Y}(\theta) - F_{M_Y}(\theta) - F_{S_Y}(\theta), \tag{2.30}$$

$$T_Z = -F_{S_Z}(\theta). \tag{2.31}$$

The expressions $\ddot{X}$ and $\ddot{Y}$ are obtained from (2.19) and (2.20) by differentiation:

$$\ddot{X} = -\omega^2[x(\theta)\cos\omega t - y(\theta)\sin\omega t] - 2\omega[\dot{x}(\theta)\sin\omega t + \dot{y}(\theta)\cos\omega t] \tag{2.32}$$
$$+\ddot{x}(\theta)\cos\omega t - \ddot{y}(\theta)\sin\omega t$$

$$\ddot{Y} = -\omega^2[x(\theta)\sin\omega t + y(\theta)\cos\omega t] - 2\omega(\dot{y}(\theta)\sin\omega t - \dot{x}(\theta)\cos\omega t) + \ddot{x}(\theta)\sin\omega t + \ddot{y}(\theta)\cos\omega t, \tag{2.33}$$

and $\ddot{x}(\theta)$, $\ddot{y}(\theta)$ are the respective derivatives of (2.8) and (2.9):

$$\ddot{x}(\theta) = -p\left[\left(\frac{\sin\theta}{(1+e\cos\theta)^2} - \frac{2e\sin\theta(e+cos\theta)}{(1+e\cos\theta)^3}\right)\dot{\theta}^2(\theta) - \frac{(e+\cos\theta)\ddot{\theta}(\theta)}{(1+e\cos\theta)^2}\right] \tag{2.34}$$

$$\ddot{y}(\theta) = -p\left[\left(\frac{\cos\theta}{(1+e\cos\theta)^2} - \frac{2e\sin^2\theta}{(1+e\cos\theta)^3}\right)\dot{\theta}^2(\theta) - \frac{\sin\theta)\ddot{\theta}(\theta)}{(1+e\cos\theta)^2}\right]. \tag{2.35}$$

In the Earth-Moon coordinate frame of Figure 1, the thrust components respectively are

$$T_x(\theta) = -T_Y\sin\omega t + T_X\cos\omega t. \tag{2.36}$$

$$T_y(\theta) = T_Y\cos\omega t + T_X\sin\omega t. \tag{2.37}$$

Given v_p we pick various values of v_a and calculate $\mathbf{T}(\theta)$ from (2.36)-(2.37) for values of θ between 0 and 2π. This is done for $0 \le e < 1$ thus finding the value of e that gives the smallest value of the maximum thrust $\mathbf{T}(\theta)$ for $0 \le \theta \le \pi$.

2.3 Computational Results

For $v_p = 5.927212$ km/sec and $v_a = 7.487672$ km/sec and values of e ranging from 0 to 0.95 the total thrust was computed in terms of θ. For values of e from 0.0 through 0.5 the maximum thrust could be found at $\theta = 0$ as depicted in Figure 2. For larger values of e the maximum thrust appears on the interval $0 \le \theta \le \pi$ between $\theta = 2$ and $\theta = 3$ as demonstrated in Figures $3-5$. Ranging over the eccentricities from 0 to 0.95 the smallest value of the maximum thrust was $T(0) = 1\ kg/sec^2$ occurring at $e = 0$, the plot appearing in Figure 6.

Setting $v_a = v_p = 7.487672$ km/sec computations show that a similar situation occurs. Again the smallest of the maximum thrusts is at $\theta = 0$ and the smallest of these again at $e = 0$.

The simulations were performed again with $v_p = 7.409016$ km/sec and $v_a = v_p/2$ with similar results. Again the maximum thrust is at $\theta = 0$ for eccentricities from 0.0 through 0.5 and occurs between $\theta = 2$ and $\theta = 3$ for eccentricities from 0.7 to 0.95. In all cases the smallest maximum thrust is at $T(0)$. Figure 7 presents a plot of the maximum thrust over eccentricity. The smallest value of the maximum thrust is about 2.35 kg/sec^2. The time to traverse from perigee to apogee for this case is calculated from (2.18) to be about 36 hours.

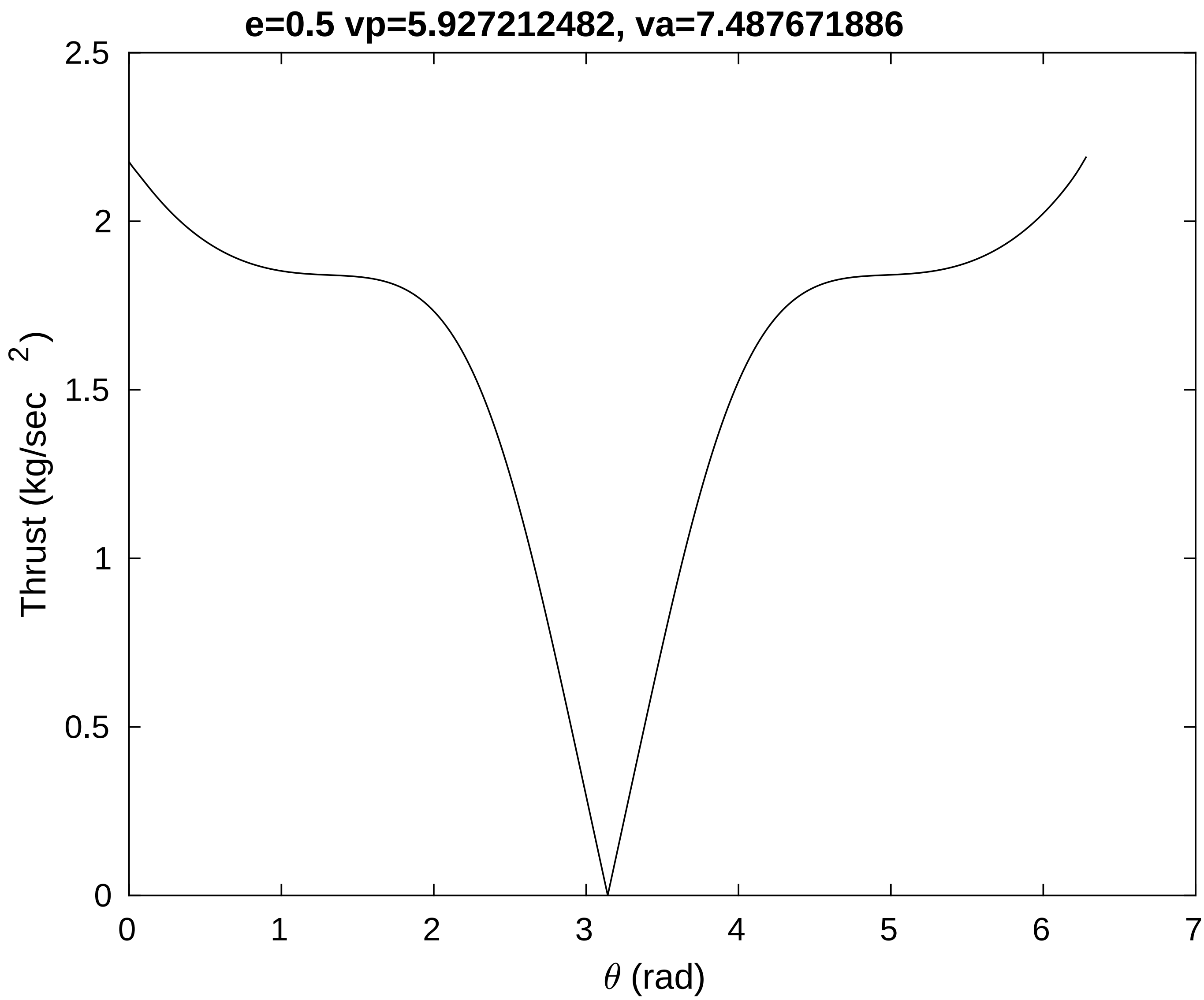

Figure 2: Thrust vs. θ, $e = 0.5$, $v_p = 5.927212$ $v_a = 7.487672$

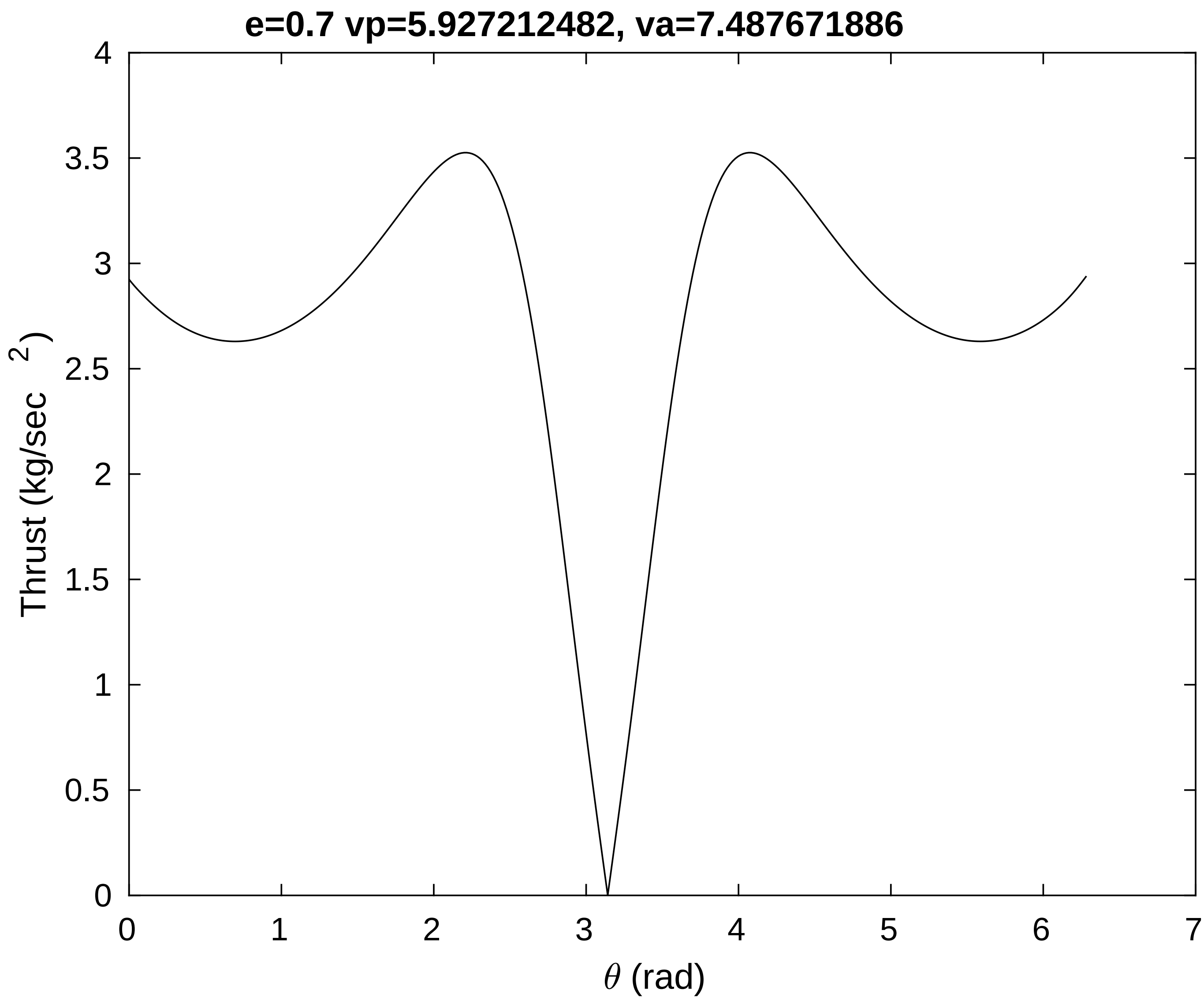

Figure 3: Thrust vs. θ, $e = 0.7$, $v_p = 5.927212$ $v_a = 7.487672$

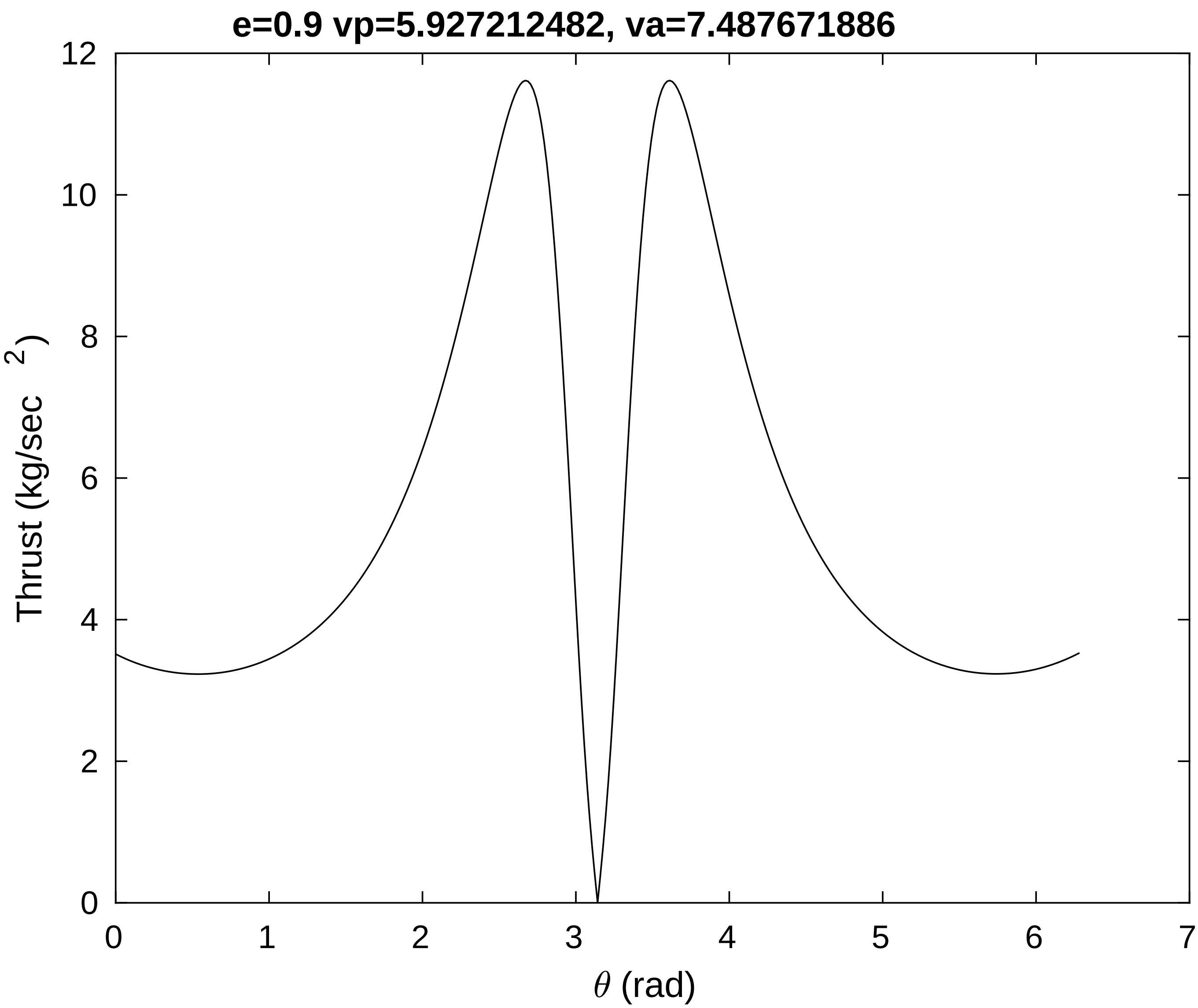

Figure 4: Thrust vs. θ, $e = 0.9$, $v_p = 5.927212$ $v_a = 7.487672$

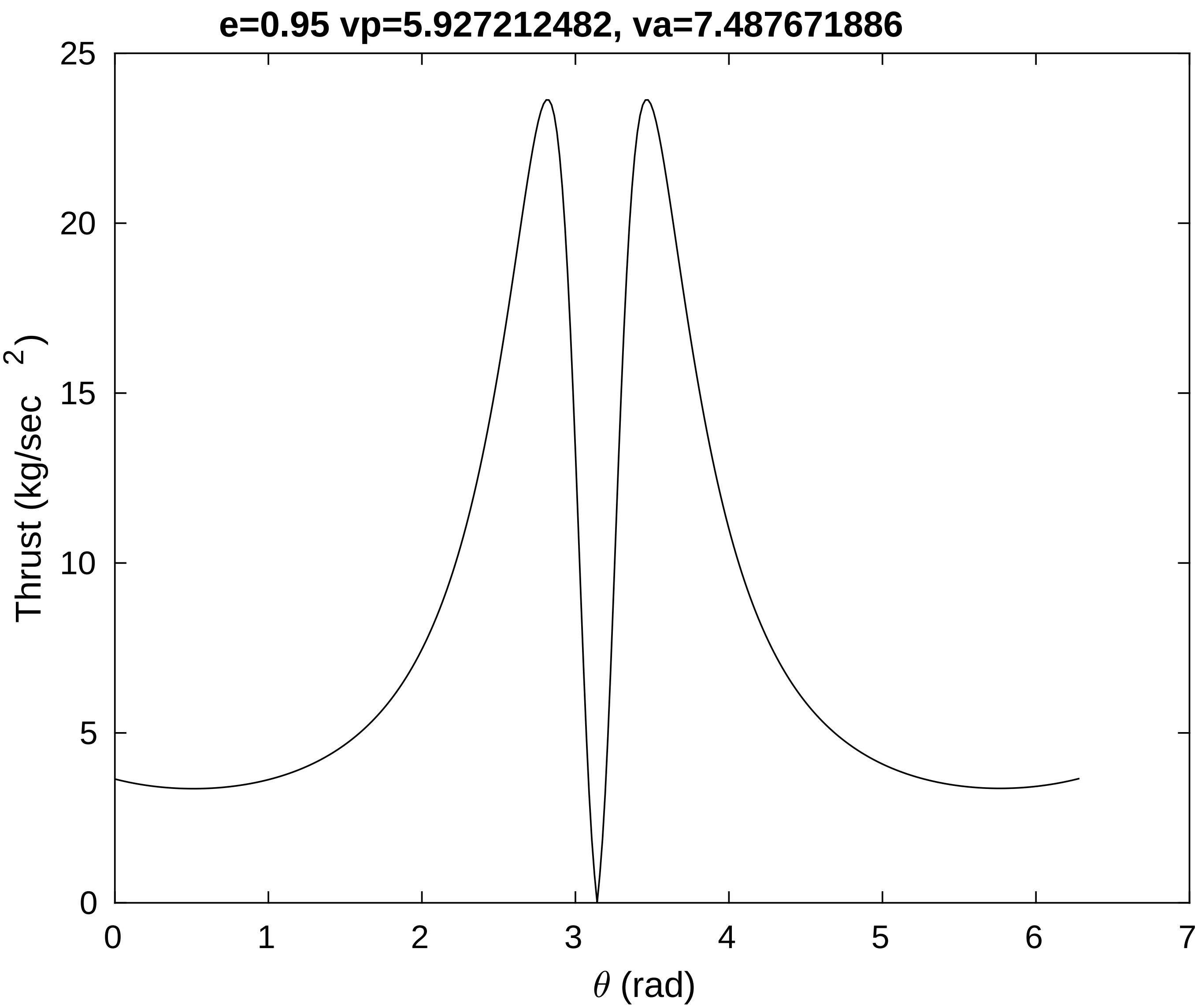

Figure 5: Thrust vs. θ, $e = 0.95$, $v_p = 5.927212$ $v_a = 7.487672$

These preliminary studies show that the least thrust requirements are obtained from a circular orbit as presented in Figure 8. There are many thrust simulations depending on different values of v_p and v_a that produce this same circle but the times t_f in proceeding from perigee to apogee will vary.

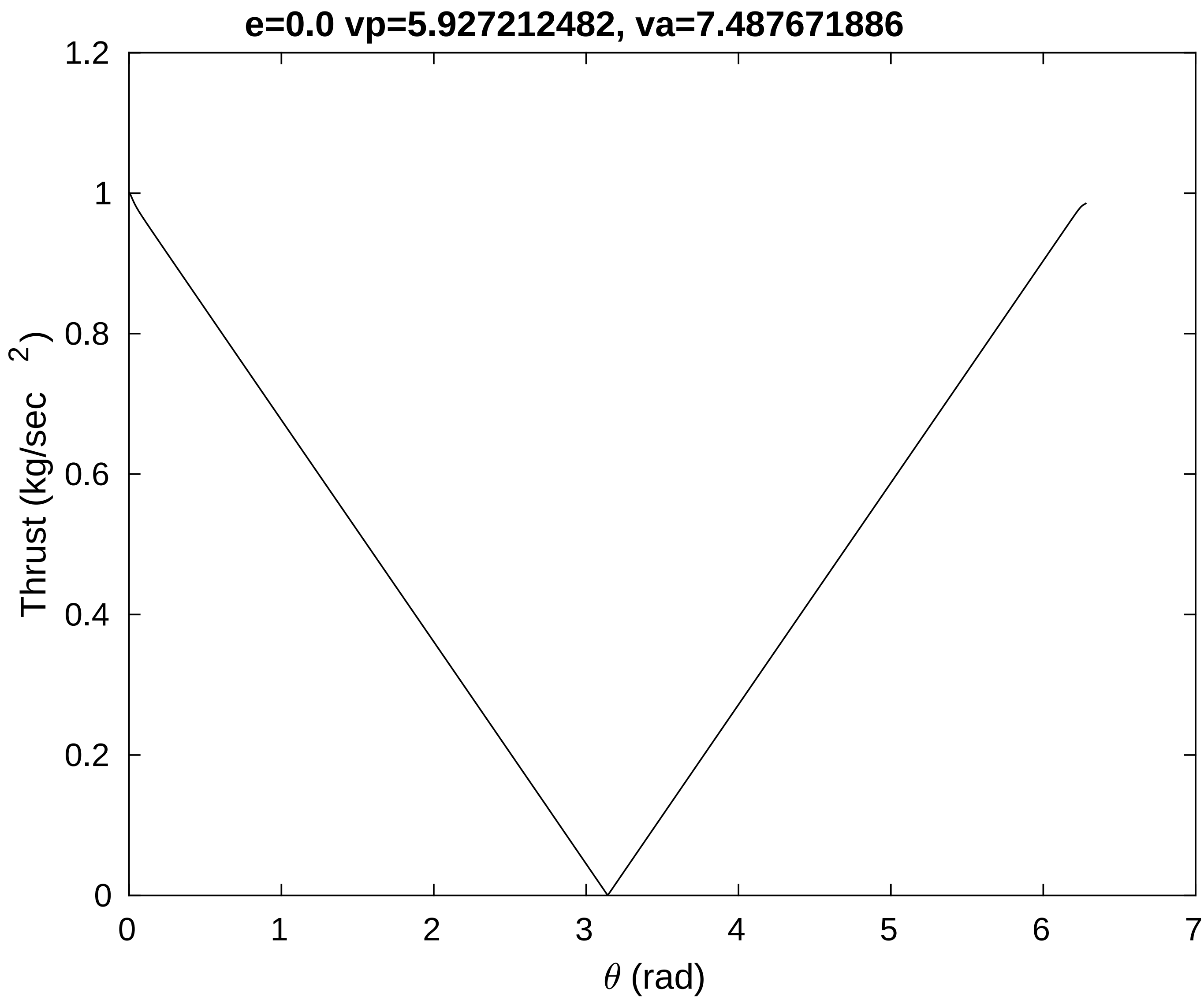

Figure 6: Thrust vs. θ, $e = 0$, $v_p = 5.927212$ $v_a = 7.487672$

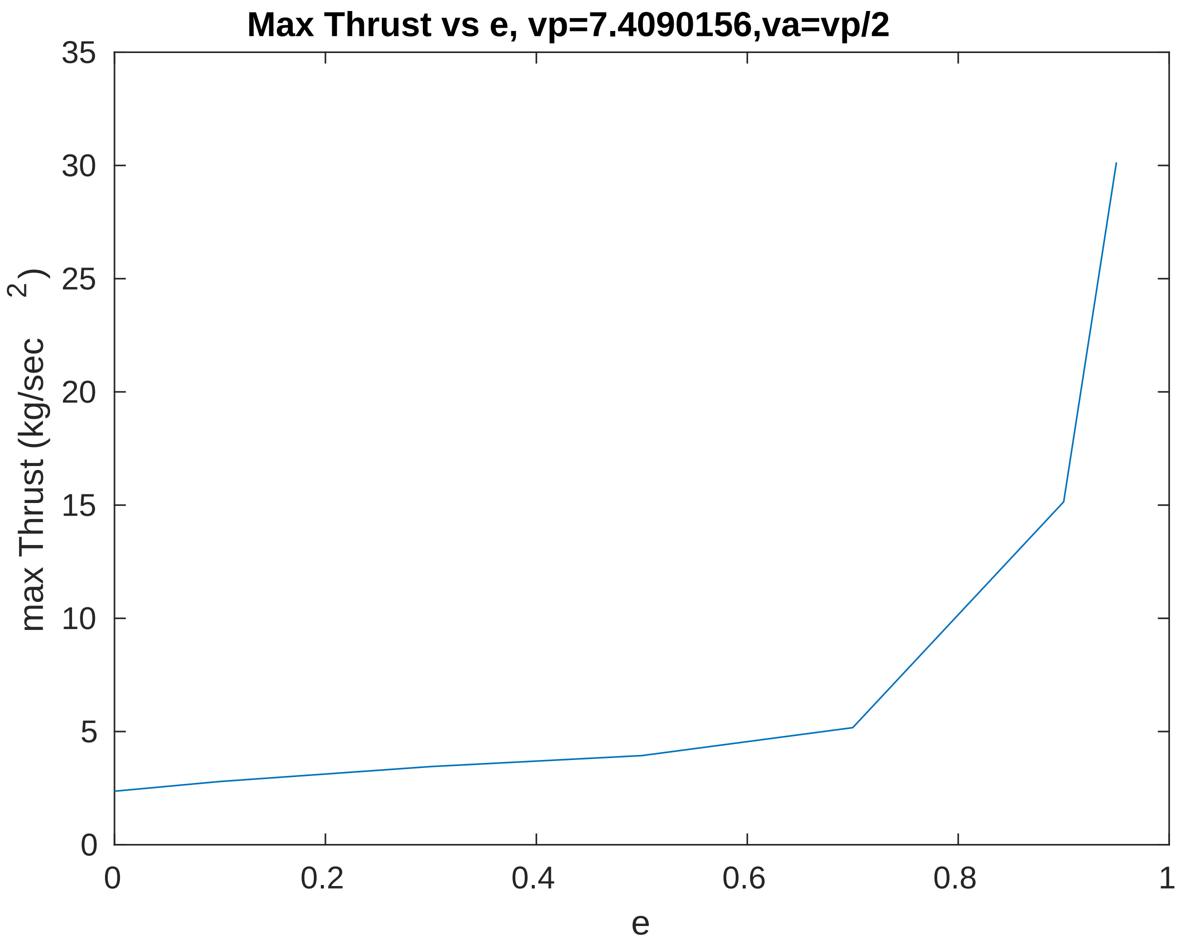

Figure 7: Maximum Thrust vs. e, $v_p = 7.409015$, $v_a = v_p/2$

2.4 Optimal Velocity at Apogee

We are also interested in determining the value of the eccentricity e that produces the optimal lunar $\Delta\, v$, the velocity between apogee and the lunar orbit or lunar surface.This is equivalent to producing a minimum of v_a

In this part of the study we are assuming that $v_p > v_a$, so it follows from (2.15) that $k > 0$ consequently (2.18) implies that the total time from perigee to to apogee is

$$t_f = \frac{\pi Arctan\sqrt{k}}{\dot{\theta}(\pi)}\sqrt{k}. \tag{2.38}$$

Solving for $\dot{\theta}(\pi)$ and using (2.13) we obtain

$$v_a = \frac{\pi}{2}(x_p + x_a)(1+e)\frac{Arctan\sqrt{k}}{t_f\sqrt{k}}. \tag{2.39}$$

This yields the following nonlinear equation in k

$$\frac{\pi(x_p + x_a)(1-e)(k+1)}{2v_p t_f}\frac{Arctan\sqrt{k}}{\sqrt{k}} = 1. \tag{2.40}$$

Given v_p and t_f we solve this equation for k in terms of e. We can then determine the velocity at apogee v_a from (2.39). For $v_p = 5.927$ km/sec and constant $t_f = 105000$ sec we present a plot of v_a versus e in Figure 9. This plot shows that also for this problem, the minimizing eccentricity occurs at $e = 0$.

3 Conclusions

This study calculated the thrust requirements to direct an Earth-Moon spacecraft into a periodic elliptical orbit. It was found that the least thrust requirements are obtained if the orbit consists of a circle through the perigee and apogee. The same result was also found to minimize the Δv between this orbit and the lunar orbit or the lunar surface. It is impressive how much the thrust requirements can be reduced by setting the eccentricity of the orbit near zero. It should be possible to devise a control system that drives the craft back to the designated orbit in order to compensate for disturbances that were not considered in the analysis. It might be argued that could one could guess that the circular orbit of the shuttle provides the optimal orbit in terms of thrust. However this orbit has a trajectory of maximum length. It follows then that the result obtained in this paper though it might be "intuitively clear" is not necessarily obvious.

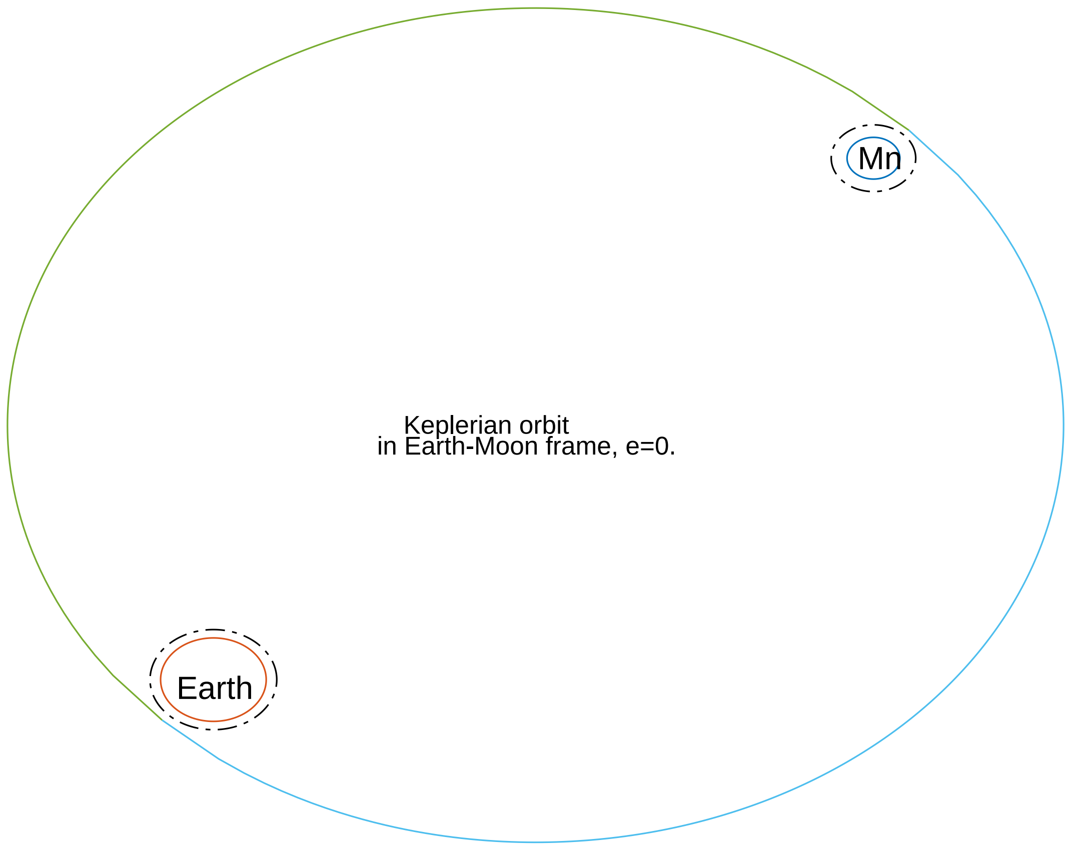

Figure 8: A schematic plot for the most efficient elliptical orbit.

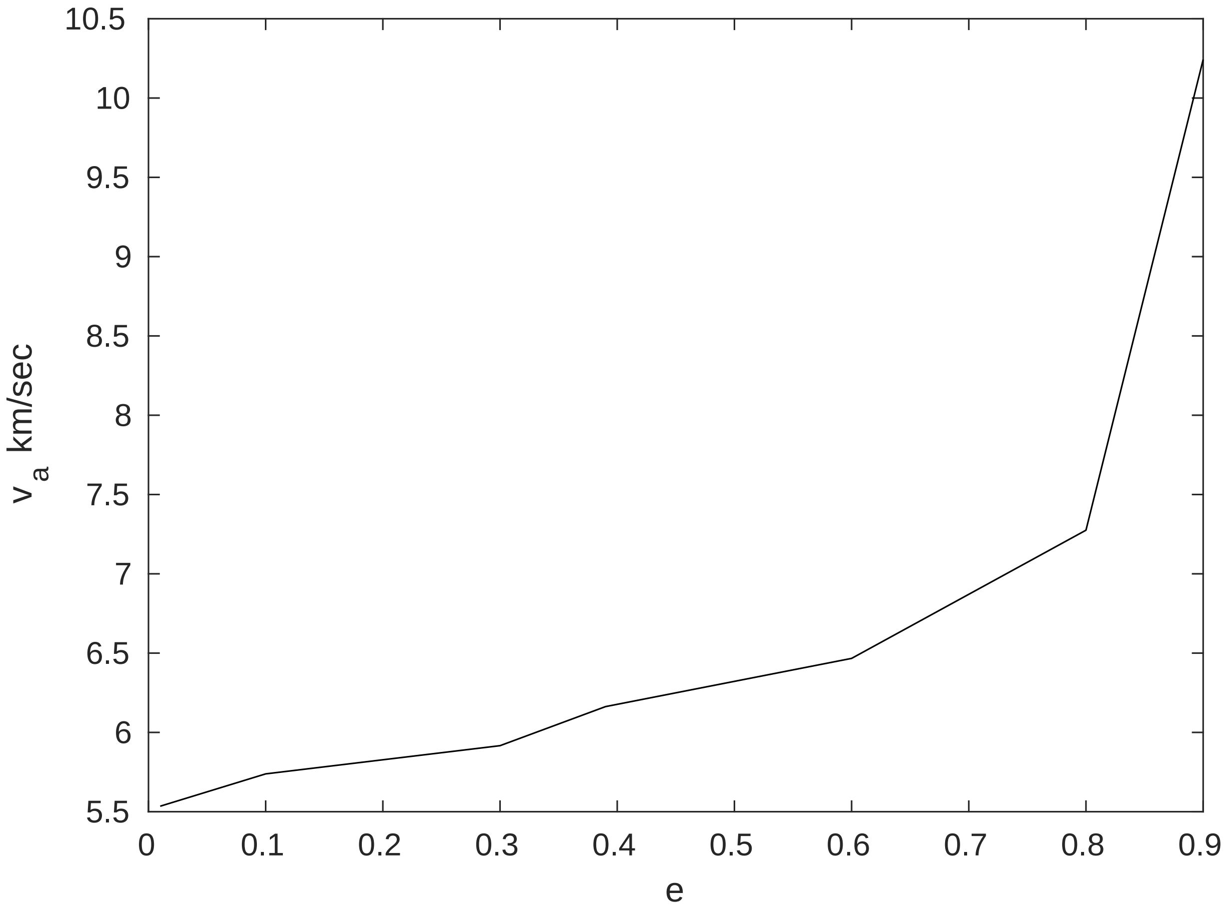

Figure 9: Velocity at Apogee vs. e: $v_p = 5.927$ km/sec, $t_f = 105000$ sec.

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